

THE DEGENERATE DISTRIBUTIVE COMPLEX IS DEGENERATE

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ABSTRACT. We prove that the degenerate part of the distributive homology of a multi-spindle is determined by the normalized homology. In particular, when the multispindle is a quandle Q , the degenerate homology of Q is completely determined by the quandle homology of Q . For this case (and generally for two-term homology of a spindle) we provide an explicit Künneth-type formula for the degenerate part. This solves the mystery in algebraic knot theory of the meaning of the degenerate quandle homology, brought over 15 years ago when the homology theories were defined, and the degenerate part was observed to be non trivial.

1. INTRODUCTION

Quandle homology [Car, CJKS, CJKLS, FRS] is build in analogy to group homology or Hochschild homology of associate structures. In the unital associative case we deal with simplicial sets (or modules) and it is a classical result that the degenerate part of a chain complex is acyclic, so homology and normalized homology are isomorphic. It is not the case for distributive structures, e.g. for quandles: we deal here only with a weak simplicial module [Prz] and the degenerate part can be not acyclic, as observed for quandles. In the concrete case of quandle homology it is proven that the homology (called the *rack homology*) splits into degenerate and normalized (called the *quandle homology*) parts [LN], but no clear general connection between degenerate and quandle part were observed.

Quandles are special case of *multispindles*, sets with a number of self-distributive operations, for which analogous homology theory exists [Prz], and with a right definition of a module over a multispindle one can also define homology with coefficients, generalizing twisted rack homology [CES]. In this paper we construct a filtration on the degenerate chain complex leading to a spectral sequence with a nice second page.

Theorem 5.4. *Let a multispindle X acts on an R -module M . Then there is a spectral sequence (E^r, ∂^r) converging to the degenerate multiterm homology $H^D(M; X)$ such that $E_{pq}^2 = H_p^N(\widehat{H}_{q-2}(M; X); X)$.*

Therefore, the degenerate homology in degree q is controlled by homology in degree less than q . This motivates to look for some recursive formula computing degenerate homology

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from the normalized one, and such a formula exists at least for (one-term) homology of spindles and (two-term) homology of quandles.

In the first case we construct an explicit isomorphism between the degenerate chain complex and its associated graded complex with respect to the filtration mentioned above, resulting in an isomorphism

$$H_n^D(M; X, \partial^\star) \cong \bigoplus_{p+q=n} \widehat{H}_{q-2}(M; X, \partial^\star) \otimes C_p^N(X)$$

for any spindle X (see Theorem 6.3). In particular, the spectral sequence mentioned above collapses at the first page. This leads to a recursive formula for degenerate homology, which can be easily solved. For instance, we obtain the following decomposition.

Corollary 6.4. *If the spindle (X, \star) is finite,*

$$H_n^D(X, \partial^\star) \cong \bigoplus_{p=1}^n \widetilde{H}_{n-p}^N(X, \partial^\star)^{\oplus \frac{|X|}{1+|X|}(|X|^p - (-1)^p)}.$$

In case of rack homology the graded associated chain complex is not isomorphic to the degenerate complex—the first page on the spectral sequence has a nonvanishing differential. Instead the degenerate chain complex $C^D(M; X)$ is isomorphic to the tensor product $\widehat{C}(M; X)[2] \otimes C^N(X)$, leading to a Künneth-type formula, see Theorem 7.1 for the exact statement. The immediate corollary is that the normalized homology controls the degenerate part.

Corollary 7.2. *Suppose a spindle homomorphism $\varphi: X \rightarrow X'$ induces an isomorphism on normalized rack homology $\varphi_*: H^N(X, \partial^R) \rightarrow H^N(X', \partial^R)$. If M is an X' -module such that the induced map $\varphi_*: H^N(M^\varphi; X, \partial^R) \rightarrow H^N(M; X', \partial^R)$ is also an isomorphism, so is $\varphi_*: H^D(M^\varphi; X, \partial^R) \rightarrow H^D(M; X', \partial^R)$.*

It is not immediate clear that the same should hold for homology of all multispindles, even for those with two nontrivial operations. The best we could achieve is that if a homomorphism of multispindles induces an isomorphism on normalized homology with coefficients in any module from a certain class, then it induces an isomorphism on degenerate homology.

Theorem 5.8. *Choose multispindles X, X' , an X' -module M , and a multispindle homomorphism $\varphi: X \rightarrow X'$ inducing an isomorphism $\varphi_*: H^N(M^\varphi; X) \rightarrow H^N(M; X')$. If it also induces an isomorphism $\varphi_*: H^N(N^\varphi; X) \rightarrow H^N(N; X')$ for any X' -module N with a vanishing compound action, then $\varphi_*: H^D(M^\varphi; X) \rightarrow H^D(M; X')$ is an isomorphism.*

We prove it by showing that such a homomorphism induces an isomorphism on the second page of the spectral sequence. The inductive argument is based on the following result on spectral sequences, which we have not encountered before.

Lemma A.8. *Assume there is a homomorphism of spectral sequences $f_{pq}^r: E_{pq}^r \rightarrow \bar{E}_{pq}^r$ such that f_{pq}^2 is an isomorphism for $q \leq N$. Then $f_{pq}^\infty: E_{pq}^\infty \rightarrow \bar{E}_{pq}^\infty$ is an isomorphism for $p + q = N$.*

Because the proof is very technical and not related directly to distributive homology, we decided to move it to the end of the paper.

Outline. The paper is organized as follows. We provide basic definitions and results on distributive and rack homology in Section 2, and homology with coefficients in modules over mutlispidles is introduced in Section 3. Most of the computation is done using a graphical calculus explained briefly in Section 4. The next three sections contain the main results of the paper: construction of the spectral sequence for degenerate homology in Section 5, and the analysis of the special cases of one-term (Section 6) and rack homology (Section 7). We provide a very brief discussion on spectral sequences in Appendix A, including the proof of the technical Lemma A.8.

2. DISTRIBUTIVE AND RACK HOMOLOGY

Definition 2.1. A *spindle* is a set X with a binary operation $\star: X \times X \rightarrow X$ that is idempotent and self-distributive from the right side, i.e. $x \star x = x$ and $(x \star y) \star z = (x \star z) \star (y \star z)$ for any $x, y, z \in X$. If the function $x \mapsto x \star y$ is invertible for any $y \in X$ we say (X, \star) is a *quandle*. By dropping the idempotent condition we obtain respectively a *shelf* and a *rack*.

The names shelf and spindle were coined by A. Crans in her PhD thesis [Cr] and are broadly used by knot theorists. The older names, used outside knot theory, are respectively a *right distributive system* and a *right distributive idempotent system*, see [Deh].

Definition 2.2. Choose a set X with a number of spindle operations \star_1, \dots, \star_r . We say $(X; \star_1, \dots, \star_r)$ is a *multispindle* if the operations are mutually distributive, i.e. $(x \star_i y) \star_j z = (x \star_j z) \star_i (y \star_j z)$ for any $x, y, z \in X$. We define *multishelves* likewise.

Remark 2.3. Because the trivial operation $x \vdash y = x$ is distributive with respect to any shelf operation [Prz, PP], one can extend any shelf (X, \star) to a mutlishelf $(X; \star, \vdash)$.

Given a spindle (X, \star) and a ring R we define $C_n(X) := R\langle X^{n+1} \rangle$ to be the R -module generated freely by all sequences $(x_n, \dots, x_0) \in X^{n+1}$. We shall write \underline{x} for such a sequence and we define $|\underline{x}| := n$. The *(one-term) distributive differential* $\partial^\star: C_n(X) \rightarrow C_{n-1}(X)$ is given as the alternating sum of *face maps* $d_i^\star: C_n(X) \rightarrow C_{n-1}(X)$:

$$(1) \quad \partial^\star = \sum_{i=0}^n (-1)^{n-i} d_i^\star,$$

$$d_i^\star(x_n, \dots, x_0) := (x_n \star x_i, \dots, x_{i+1} \star x_i, x_{i-1}, \dots, x_0).$$

The unusual sign convention for ∂^\star is the result of enumerating elements in a sequence \underline{x} from right to left, contrary to the standard practice [CJKLS, CJKS, Prz, PS]. We check the presimplicial relation $d_i^\star d_j^\star = d_{j-1}^\star d_i^\star$ for $i > j$ (see also Example 4.1), from which it follows $(\partial^\star)^2 = 0$. We call the homology of this chain complex the *(one-term) distributive homology* of X .

Given a multishelf $(X; \star_1, \dots, \star_r)$ one can check that $\partial^{\star_i} \partial^{\star_j} + \partial^{\star_j} \partial^{\star_i} = 0$. This guarantees that any linear combination $\partial = \sum_{k=1}^r a_k \partial^{\star_k}$ is a differential on $C(X)$, which we call the *multiterm distributive differential* with weights (a_1, \dots, a_r) . Of particular interest is the case of the *rack differential* $\partial^R = \partial^+ - \partial^\star$, where $x \vdash y := x$ [CJKS, FRS]. Notice that our rack homology is shifted by one comparing to the definition due to Fenn, Rourke, and Sanderson. The main reason is that we wanted to deal with a pre-simplicial category while they chose the convention of a pre-cubic category.

Remark 2.4. We shall write $C(X)$ for the chain complex and $H(X)$ for its homology if we do not want to specialize the differential. Otherwise, the notation $C(X, \partial^\star)$, $H(X, \partial^R)$, etc. will be used.

Definition 2.5. Let X be a multispindle. The *degenerate chain complex* $C^D(X)$ is the subcomplex of $C(X)$ generated by sequences \underline{x} with repetitions, i.e. $x_i = x_{i+1}$ for some i . The quotient $C^N(X) := C(X)/C^D(X)$ is called the *normalized complex* of X . Homology of the complexes are called respectively *degenerate* and *normalized*, and they are denoted by $H^D(X)$ and $H^N(X)$.

Remark 2.6. In case of quandles, the normalized homology is usually referred to as the *quandle homology* and is written as $H^Q(X)$, see [CJKS].

The following theorem was first proven for the rack homology of quandles [LN], and then extended to the one- and multiterm case [Prz].

Theorem 2.7 ([LN, Prz]). *Given a multispindle X , the following short exact sequence*

$$(2) \quad 0 \longrightarrow C^D(X) \longrightarrow C(X) \longrightarrow C^N(X) \longrightarrow 0.$$

splits in a canonical way. In particular, $H(X) \cong H^N(X) \oplus H^D(X)$.

In case of rack homology sequences \underline{x} with late repetitions (i.e. $x_i = x_{i-1}$ for some $i < n$ in (x_n, \dots, x_0)) form a chain subcomplex $C^L(X, \partial^R) \subset C^D(X, \partial^R)$, which is a direct summand. It is called the *late degenerate complex* in [LN].

Theorem 2.8 ([LN]). *There is a short exact sequence that splits canonically*

$$(3) \quad 0 \longrightarrow C^L(X, \partial^R) \longrightarrow C^D(X, \partial^R) \longrightarrow C^N(X, \partial^R)[1] \longrightarrow 0.$$

In particular, $H_n^N(X, \partial^R)$ is a direct summand of $H_{n+1}^D(X, \partial^R)$.

The number in brackets indicates a homological shift: $C[k]_n := C_{n-k}$.

In the general case of multiterm homology $C^L(X)$ is a subcomplex of $C^D(X)$ if and only if the sum of all weights is zero, in which case the proof from [LN] of the theorem above holds. We shall generalize it to homology with coefficients in the next section.

3. HOMOLOGY WITH COEFFICIENTS

The following definition is motivated by the notion of an X -set, introduced by S. Kamada.

Definition 3.1. An *action* of a spindle (X, \star) on an R -module M is a function $\star: M \times X \rightarrow M$ that is linear in the first variable and

$$(4) \quad (m \star x) \star y = (m \star y) \star (x \star y)$$

for any $m \in M$ and $x, y \in X$. If (X, \star) is a quandle we require also that it acts by automorphisms of M , i.e. the function $m \mapsto m \star x$ is invertible for every $x \in X$.

An R -module M carrying an action of a spindle X will be often called an X -module.

Example 3.2. Consider a spindle (X, \vdash) with the trivial operation $x \vdash y = x$. Then the actions on M of all elements of X commute:

$$(5) \quad (m \vdash x) \vdash y = (m \vdash y) \vdash (x \vdash y) = (m \vdash y) \vdash x.$$

Hence, X -modules are precisely modules over the polynomial algebra $\mathbb{Z}[X]$ with as many variables as there are elements in X .

We generalize Definition 3.1 to multispindles in a natural way.

Definition 3.3. An action of a multispindle $(X; \star_1, \dots, \star_r)$ on an R -module M consists of functions $\star_i: M \times X \rightarrow M$ linear in the first variable, such that

$$(6) \quad (m \star_i x) \star_j y = (m \star_j y) \star_i (x \star_j y)$$

for any $m \in M$, $x, y \in X$ and $i, j = 1, \dots, r$.

Given a multispindle $(X; \star_1, \dots, \star_r)$, which acts on M , choose $a_1, \dots, a_r \in R$. We define the *compound action* of X on M with weights (a_1, \dots, a_r) as the linear combination

$$(7) \quad (m, x) \mapsto m \cdot x := \sum_{i=1}^r a_i (m \star_i x).$$

A direct computation shows the compound action is distributive with respect to the action of X on M , i.e. $(m \cdot x) \star_i y = (m \star_i y) \cdot (x \star_i y)$ for $i = 1, \dots, r$.

Example 3.4. Every multispindle X acts on any module M trivially, $m \star_i x := m$, in which case the compound action is the multiplication by the sum of weights.

Example 3.5. A multispindle X acts on the distributive chain complex $C(X)$ by acting on each element of a sequence: $(x_n, \dots, x_0) \star_i y := (x_n \star_i y, \dots, x_0 \star_i y)$. This action descends to homology and it was observed in [PP] that $\alpha \cdot w = 0$ for any homology class $\alpha \in H(X)$ and $w \in X$. Indeed, $\alpha \cdot w = \partial h^w + h^w \partial$ for $h^w(\underline{x}) := (-1)^{|\underline{x}|}(\underline{x}, w)$.

Let a multispindle X acts on an R -module M . We define the *homology with coefficients in* M by setting $C_n(M; X) := M \otimes C_n(X)$ and $d_i^{\star k}(m \otimes \underline{x}) := (m \star_k x_i) \otimes d_i^{\star k} \underline{x}$. Clearly, $C(X) = C(R; X)$ with the trivial action of X on R . As before, we have the degenerate subcomplex with $C_n^D(M; X) := M \otimes C_n^D(X)$ and the normalized one $C^N(M; X) := C(M; X)/C^D(M; X)$.

Proposition 3.6. *The short exact sequence splits canonically*

$$(8) \quad 0 \longrightarrow C^D(M; X) \longrightarrow C(M; X) \longrightarrow C^N(M; X) \longrightarrow 0.$$

In particular, $H(M; X) \cong H^N(M; X) \oplus H^D(M; X)$.

Proof. Let $\alpha: C^N(X) \rightarrow C(X)$ be the splitting map from Theorem 2.7. Because each $C_n(X)$ is a free R -module, $C_n^N(M; X) = M \otimes C_n^N(X)$ and an easy computation shows the map $\alpha^M := \text{id} \otimes \alpha$ splits the sequence (8). \square

The case of modules M with a vanishing compound action is very special. As before let $C_n^L(M; X) := M \otimes C_n^L(X)$ be spanned by late degenerate sequences.

Proposition 3.7. *If the compound action of X annihilates M , then $C^L(M; X)$ is a subcomplex of $C^D(M; X)$ and there is an isomorphism $s: C^N(M; X)[1] \rightarrow C^D(M; X)/C^L(M; X)$ given by the formula $s(m \otimes (x_n, \dots, x_0)) = (-1)^n m \otimes (x_n, x_n, \dots, x_0)$.*

Proof. Vanishing of the compound action implies vanishing of the compound face map $d_n: C_n(M; X) \rightarrow C_{n-1}(M; X)$, $d_n = \sum_k a_k d_n^{\star k}$. Hence, $C^L(M; X)$ is a subcomplex of $C^D(M; X)$. For s to be an isomorphism we have to check it is a chain map, which follows from the equality $d_n^{\star k} = d_{n-1}^{\star k}$ on $C_n^D(M; X)/C^L(M; X)$. Indeed, every sequence in the quotient $C^D(M; X)/C^L(M; X)$ begins with a repetition. \square

Remark 3.8. One can follow the proof of Theorem 2.8 from [LN] to see that $C^L(M; X)$ is a direct summand of $C^D(M; X)$.

Remark 3.9. A multispindle X acts naturally on its homology with coefficients. The compound action vanishes, which is proven using the homotopy $h^w(m \otimes \underline{x}) := (-1)^{|\underline{x}|} m \otimes (\underline{x}, w)$.

The one- and multiterm chain complexes can be augmented with $C_0(X) \xrightarrow{\epsilon} R$ sending each generator to $\epsilon(x) := 1$. This new chain complex and its homology are written as $\tilde{C}(X)$ and $\tilde{H}(X)$ respectively. We redefine it for complexes with coefficients in an X -module by appending to the chain complex $C(M; X)$ the linearized compound action $C_0(M; X) = M \otimes R\langle X \rangle \rightarrow M$. We write $\hat{C}(M; X)$ and $\hat{H}(M; X)$ for this chain complex and its homology respectively.



FIGURE 1. A string diagram seen as a function $X^3 \longrightarrow X^2$. The pictures to the right explain how labels propagate through a crossing and where the motivation comes from.

Remark 3.10. $\hat{H}_n(M; X) = H_n(M; X)$ for $n \geq 0$ and $\hat{H}_{-1}(M; X) = M$ when the compound action vanishes. In particular, $\hat{H}(R; X)$ is different from $\tilde{H}(R; X)$ in case of rack homology.

We construct the augmented normalized complex $\hat{C}^N(M; X)$ with homology $\hat{H}^N(M; X)$ likewise. It is worth to notice that Proposition 3.6 still holds for augmented complexes.

4. GRAPHICAL CALCULUS

Consider a picture in a plane consisting of a number of strands originating on a horizontal line and going upwards to another horizontal line. The strands can cross with each other¹ and a single strand can terminate in a dot but there are no turnbacks. Given a spindle (X, \star) we can interpret such a picture as a map $X^n \longrightarrow X^m$, where n and m are numbers of endpoints of strands at the lower and upper horizontal lines respectively. Namely, decorate the n endpoints on the lower line with elements of X and propagate the labels upwards along the strands. A label is forgotten at a terminal dot, and each time we encounter a crossing the right label propagates with no change but the left one is replaced by the product of both, see Fig. 1.

Different pictures can encode the same map. For instance, far away crossings and terminal dots commute:

$$(9) \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \cdots \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} \cdots \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \cdots \bullet = \begin{array}{c} \diagdown \\ \diagup \end{array} \cdots \bullet \quad \text{etc.}$$

and a terminal dot can be pulled through a crossing from the left hand side but not from the right one:

$$(10) \quad \begin{array}{c} \bullet \\ \diagdown \\ \diagup \end{array} = \begin{array}{c} \bullet \\ \diagup \\ \diagdown \end{array}, \quad \text{but} \quad \begin{array}{c} \bullet \\ \diagup \\ \diagdown \end{array} \neq \begin{array}{c} \bullet \\ \diagdown \\ \diagup \end{array}.$$

The most interesting are the following two relations

$$(11) \quad \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} | \\ | \end{array} \quad \text{and} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array},$$

¹ These are real crossings on a plane but we can interpret them with no harm as negative crossings.

which are equivalent to the axioms of a spindle. The first one holds only when both inputs are labeled with the same element, and it visualizes the idempotency axiom. The right picture is the famous Reidemeister III move, also called the *braid* or the *Yang-Baxter relation*. It holds for any input and is equivalent to the self-distributivity of $\star: X \times X \rightarrow X$.

If X acts on a module M , add a wall to the left of the picture—it can be labeled with elements $m \in M$. A strand can terminate on the wall, which corresponds to the action map $M \times X \rightarrow M$. The condition for an action of X translates as absorbing a crossing by the wall:

$$(12) \quad \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. A strand starts from the wall, goes right, then curves down and right, crossing over another strand that starts from the wall and goes straight right.} \end{array} = \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. A strand starts from the wall, goes right, then curves down and right, crossing under another strand that starts from the wall and goes straight right.} \end{array}$$

Example 4.1. The i -th face map $d_i^*: C_n(M; X) \rightarrow C_{n-1}(M; X)$ can be visualized graphically by pulling the i -th strand all the way to the left, and terminating it at the wall. Then the presimplicial relation follows from an easy deformation of pictures. For instance,

$$d_i^* := \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. A strand starts from the wall, goes right, then curves down and right, crossing over another strand that starts from the wall and goes straight right. The strand that crosses over is labeled 'n', the strand that crosses under is labeled 'i', and the wall is labeled '0'.} \end{array}$$

$$(13) \quad d_i^* d_j^* = \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. Two strands start from the wall, go right, then curve down and right, crossing over each other. The left strand is labeled 'n', the right strand is labeled 'j', and the wall is labeled '0'.} \end{array} = \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. Two strands start from the wall, go right, then curve down and right, crossing under each other. The left strand is labeled 'n', the right strand is labeled 'j', and the wall is labeled '0'.} \end{array} = \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. Two strands start from the wall, go right, then curve down and right, crossing over each other. The left strand is labeled 'n', the right strand is labeled 'j', and the wall is labeled '0'.} \end{array} = d_{j-1}^* d_i^*,$$

when $j > i$.

Example 4.2. In the case of the trivial operation $x \vdash y = x$, we can use a simpler picture for the face map d_i^+ : the i -th strand is immediately terminated with a dot. The presimplicial relation follows trivially, and in the mixed case we use (10). For instance, when $j > i$, one computes

$$d_i^+ := \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. A strand starts from the wall, goes right, then curves down and right, crossing over another strand that starts from the wall and goes straight right. The strand that crosses over is labeled 'n', the strand that crosses under is labeled 'i', and the wall is labeled '0'.} \end{array}$$

$$(14) \quad d_i^* d_j^+ = \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. Two strands start from the wall, go right, then curve down and right, crossing over each other. The left strand is labeled 'n', the right strand is labeled 'j', and the wall is labeled '0'.} \end{array} = \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. Two strands start from the wall, go right, then curve down and right, crossing under each other. The left strand is labeled 'n', the right strand is labeled 'j', and the wall is labeled '0'.} \end{array} = d_{j-1}^+ d_i^*.$$

We shall often restrict to sequences with a repetition at a certain place, which will be visualized by a band joining two strands: it forces its two edges to carry the same label. For instance, we can visualize generators of $C^D(M; X)$ by vertical lines with a band at some position:

$$(15) \quad C_n^D(M; X) := \left\langle \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. A strand starts from the wall, goes right, then curves down and right, crossing over another strand that starts from the wall and goes straight right. The strand that crosses over is labeled 'n', the strand that crosses under is labeled 'i', and the wall is labeled '0'.} \end{array} : 0 \leq i < n \right\rangle,$$

and the idempotency axiom can be rewritten without specifying labels at the input as

$$(16) \quad \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. A strand starts from the wall, goes right, then curves down and right, crossing over another strand that starts from the wall and goes straight right. The strand that crosses over is labeled 'n', the strand that crosses under is labeled 'i', and the wall is labeled '0'.} \end{array} = \begin{array}{c} \text{Diagram: A vertical wall (hatched line) on the left. A strand starts from the wall, goes right, then curves down and right, crossing under another strand that starts from the wall and goes straight right. The strand that crosses over is labeled 'n', the strand that crosses under is labeled 'i', and the wall is labeled '0'.} \end{array}.$$

Strands that are not joined by a band can still carry the same label, however. Finally, the map $s_p: C_p(M; X) \rightarrow C_{p+1}(M; X)$ can be visualized by thickening the left most strand

into a band

$$(17) \quad s_p = (-1)^p \begin{array}{c} \text{diagram with } p \text{ strands, a purple band on the } p\text{-th strand, and } i \text{ strands to the right} \\ p \quad i \quad 0 \end{array}.$$

Remark 4.3. We shall use the same graphical calculus for multispindles without mixing different operations, i.e. we shall always use one operation to interpret a diagram.

5. A FILTRATION OF THE DEGENERATE COMPLEX

Let $\mathcal{F}_n^p \subset C_n^D(M; X)$ be the submodule generated by elements $m \otimes \underline{x}$, where the sequence \underline{x} has a repetition at position p or closer to the right, i.e. $x_i = x_{i+1}$ for some $i \leq p$. Diagrammatically,

$$(18) \quad \mathcal{F}_n^p := \left\langle \begin{array}{c} \text{diagram with } n \text{ strands, a purple band on the } i\text{-th strand, and } 0 \text{ strands to the right} \\ n \quad i \quad 0 \end{array} : 0 \leq i \leq p \right\rangle.$$

Proposition 5.1. *Each \mathcal{F}^p is a chain subcomplex of $C^D(M; X)$, and all together they form a filtration of $C^D(M; X)$, i.e. $\mathcal{F}^p \subset \mathcal{F}^{p+1}$ for every p , and $\bigcup_{p \geq 0} \mathcal{F}^p = C^D(M; X)$.*

Proof. The two conditions for a filtration follow directly from the definition of \mathcal{F}_n^p . To show that \mathcal{F}^p is closed under the differential, choose a sequence \underline{x} with a repetition at position i . Then for any operation \star_k on X the faces $d_i^{\star_k}(m \otimes \underline{x}) = d_{i+1}^{\star_k}(m \otimes \underline{x})$ cancel each other, whereas the other faces have repetitions at positions i or $i - 1$:

$$(19) \quad \begin{array}{c} \text{diagram with } j \text{ strands, a purple band on the } i\text{-th strand, and } 0 \text{ strands to the right} \\ j \quad i \quad 0 \end{array} \quad \text{or} \quad \begin{array}{c} \text{diagram with } i-1 \text{ strands, a purple band on the } i\text{-th strand, and } 0 \text{ strands to the right} \\ i-1 \quad i \quad 0 \end{array}.$$

□

Because the filtration is given on generators, the quotients $\mathcal{F}^p / \mathcal{F}^{p-1}$ are easy to understand: each is generated by elements $m \otimes \underline{x}$, where the sequence \underline{x} has its first repetition occurring at position p , when looking from the right hand side. As $d_i^{\star_k}(m \otimes \underline{x}) \in \mathcal{F}^{p-1}$ if $i < p$, only higher faces survive.

Corollary 5.2. *There is an isomorphism of chain complexes*

$$(20) \quad \mathcal{F}^p / \mathcal{F}^{p-1} \cong \widehat{C}(M; X)[p+2] \otimes C_p^N(X),$$

where on the right hand side the differential acts only on the first factor.

Proof. The face maps $d_i^{\star_k}$ vanish on the quotient for $i < p$, and since $d_p^{\star_k}$ cancels $d_{p+1}^{\star_k}$,

$$(21) \quad \begin{array}{c} \text{diagram with } n \text{ strands, a purple band on the } p\text{-th strand, and } 0 \text{ strands to the right} \\ n \quad p \quad 0 \end{array} = \begin{array}{c} \text{diagram with } n \text{ strands, a purple band on the } p\text{-th strand, and } 0 \text{ strands to the right} \\ n \quad p \quad 0 \end{array}.$$

Hence, the repetition splits \underline{x} into two parts: the left one, which consists only of the wall (thence the augmented complex in (20)), and the right one (including the p -th strand), which has no repetition. The latter is preserved by the differential. □

Recall that the *graded associated* chain complex $\text{gr } C^D(M; X)$ with respect to the filtration \mathcal{F}^p is the direct sum of quotients $\mathcal{F}^p/\mathcal{F}^{p-1}$.

Corollary 5.3. *Let $(X; \star_1, \dots, \star_r)$ be a multispindle acting on a module M . Then*

$$(22) \quad H_n(\text{gr } C^D(M; X)) \cong \bigoplus_{p+q=n} \widehat{H}_{q-2}(M; X) \otimes C_p^N(X).$$

The filtration \mathcal{F}^p leads to a spectral sequence (see Appendix A) computing the degenerate homology of X , and the corollary above shows its first page. Its second page is the normalized homology of X with coefficients in the augmented homology.

Theorem 5.4. *Let a multispindle X acts on an R -module M . Then there is a spectral sequence (E^r, ∂^r) converging to the degenerate multiterm homology $H^D(M; X)$ such that $E_{pq}^2 = H_p^N(\widehat{H}_{q-2}(M; X); X)$.*

Proof. Due to Corollary 5.2 there is an isomorphism

$$\text{id} \otimes s_p: \widehat{H}_{q-2}(M; X) \otimes C_p^N(X) \xrightarrow{\cong} E_{pq}^1,$$

so that the differential $\partial_{pq}^1: E_{pq}^1 \rightarrow E_{p-1,q}^1$ computes homology of $sC^N(X) \cong C^D(X)/C^L(X)$ with coefficients in $\widehat{H}_{q-2}(M; X)$. Since the compound action of X on $\widehat{H}_{q-2}(M; X)$ vanishes,

$$s: C^N(\widehat{H}_{q-2}(M; X); X) \rightarrow sC^N(\widehat{H}_{q-2}(M; X); X)$$

is actually a chain map, thence an isomorphism. \square

Corollary 5.5. *Assume $\widehat{H}^N(M; X) = 0$. Then degenerate homology vanishes. In particular, if the augmented homology of a spindle is trivial, so is its degenerate homology.*

Proof. Suppose $H_i^D(M; X) = 0$ for $i < N$. Then $E_{pq}^r = H_p^N(\widehat{H}_{q-2}(M; X); X) = 0$ for $q < N + 2$. In particular, $E_{pq}^\infty = 0$ when $p + q = N$, which implies $H_N^D(M; X) = 0$. \square

Corollary 5.6. *Assume the augmented quandle homology (i.e. $\partial = \partial^{-1} - \partial^*$) of a spindle X is trivial. Then $H_n^D(R; X) = R$ for $n > 0$.*

Proof. The compound action on R vanishes, so that $\widehat{H}_0^N(R; X) = \widehat{H}_{-1}^N(R; X) = R$ and $\widehat{H}_p^N(R; X) = H_p^N(R; X) = 0$ for $p > 0$. Hence, $E_{1,0}^2 = 0$ and $E_{0,1}^2 = R$, which implies $H_1^D(R; X) = E_{0,1}^2 = R$. Suppose by induction that $H_i^D(R; X) = R$ for $i = 1, \dots, N - 1$. Then $E_{pq}^2 = H_p^N(R; X)$ for $q < N + 2$, which is zero except $E_{0,q}^2 = H_0(R; X) = R$. Again, it must be $H_N^D(R; X) = E_{0,N}^2 = R$. \square

Choose a homomorphism of spindles $\varphi: X \rightarrow X'$ and an X' -module M . Then φ induces an action of X on M , $m \star_i x := m \star_i \varphi(x)$, and we shall write M^φ for this X -module.

Corollary 5.7. *Suppose a homomorphism of multispindles $\varphi: X \rightarrow X'$ induces isomorphisms on homology groups $\varphi_*: H^N(\widehat{H}_q(M^\varphi; X); X) \rightarrow H^N(\widehat{H}_q(M; X'); X')$ for any q . Then $\varphi_*: H^D(M^\varphi; X) \rightarrow H^D(M; X')$ is an isomorphism.*

Assume that $\varphi: X \rightarrow X'$ given as above induces also an isomorphism on normalized homology $\varphi_*: H^N(M^\varphi; X) \rightarrow H^N(M; X')$. Then $H(M^\varphi; X) \cong H(M; X')^\varphi$ as X -modules and we can restate the assumptions on φ requiring that it induces an isomorphism on normalized homology with coefficient in any module with a vanishing compound action. Since only $\widehat{H}_k(M; X)$ with $k \leq n - 2$ appear in E_{pq}^2 with $p + q = n$, one should be able to recover the original assumption of Corollary 5.7 by an induction argument. This is the idea underlying the following theorem.

Theorem 5.8. *Choose multispindles X, X' , an X' -module M , and a multispindle homomorphism $\varphi: X \rightarrow X'$ inducing an isomorphism $\varphi_*: H^N(M^\varphi; X) \rightarrow H^N(M; X')$. If it also induces an isomorphism $\varphi_*: H^N(N^\varphi; X) \rightarrow H^N(N; X')$ for any X' -module N with a vanishing compound action, then $\varphi_*: H^D(M^\varphi; X) \rightarrow H^D(M; X')$ is an isomorphism.*

Proof. Consider the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^\varphi[-1] & \longrightarrow & \widehat{C}(M^\varphi; X) & \longrightarrow & C(M^\varphi; X) \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \widehat{\varphi}_* & & \downarrow \varphi_* \\ 0 & \longrightarrow & M[-1] & \longrightarrow & \widehat{C}(M; X') & \longrightarrow & C(M; X') \longrightarrow 0 \end{array}$$

Due to the 5-lemma, $\widehat{\varphi}_*$ induces an isomorphism on homology if and only if so does φ_* .

Denote by E^r and \bar{E}^r the spectral sequences for $H^D(M^\varphi; X)$ and $H^D(M; X')$. Since $\widehat{H}_k(M^\varphi; X) = \widehat{H}_k^N(M^\varphi; X)$ and $\widehat{H}_k(M; X') = \widehat{H}_k^N(M; X')$ for $k < 1$, we immediately see that $f_{pq}^2: E_{pq}^2 \rightarrow \bar{E}_{pq}^2$ is an isomorphism for $q < 3$ (homology is a module with a vanishing compound action). It follows now from Lemma A.8 that f_{pq}^∞ is an isomorphism if $p + q = 1$, so that $H_1^D(M^\varphi; X) \cong H_1^D(M; X')$. Hence, f_{pq}^2 is an isomorphism for $q < 4$ and similarly we get $H_2^D(M^\varphi; X) \cong H_2^D(M; X')$. Use induction to finish the proof. \square

In the next two sections we shall strengthen this result for the one-term and rack homology, obtaining recursive formulas for degenerate homology in terms of the normalized one.

6. DEGENERATE SPINDLE HOMOLOGY

The filtration \mathcal{F}^p of $C^D(M; X)$ is given on generators, so that the chain groups $\text{gr } C_n^D(M; X)$ and $C_n^D(M; X)$ are naturally isomorphic. We shall identify them together. However, this identification is not compatible with differentials, and our goal is to find correcting terms to obtain a chain map $f: \text{gr } C_n^D(M; X) \rightarrow C_n^D(M; X)$ that is *filtered*, i.e. we require f to send $\text{gr}_p C^D(M; X)$ into \mathcal{F}^p . The following lemma is a classical result from the theory of filtered modules [M].

Lemma 6.1. *Let C and D be chain complexes with filtrations $\mathcal{F}^p C$ and $\mathcal{F}^p D$ respectively, and let $f: C \rightarrow D$ be a filtered chain map, i.e. $f(\mathcal{F}^p C) \subset \mathcal{F}^p D$ for any p . If $\text{gr } f: \text{gr } C \rightarrow \text{gr } D$ is an isomorphism, so is f .*

Corollary 6.2. *A filtered chain map $f: \text{gr } C^D(M; X) \longrightarrow C^D(M; X)$ is an isomorphism if and only if $\text{gr}_p C^D(M; X) \xrightarrow{f} \mathcal{F}^p \xrightarrow{pr} \mathcal{F}^p / \mathcal{F}^{p-1}$ is an isomorphism for every p .*

We define $\mathcal{E}: \text{gr } C^D(M; X) \longrightarrow C^D(M; X)$ by correcting the identity homomorphisms with lower order terms. Namely, the component $\mathcal{E}^p: \text{gr}_p C^D(M; X) \longrightarrow \mathcal{F}^p$ is given by the following picture

$$(23) \quad \mathcal{E}^p := \begin{array}{c} \text{Diagram with } n \text{ vertical lines on the left, } p \text{ vertical lines in the middle, and } 0 \text{ vertical lines on the right. A box labeled } u^p \text{ is connected to the } p \text{ lines. The box has a purple shaded bottom-left corner.} \end{array},$$

where the map $u^p: C_{p+1}(X) \longrightarrow C_{p+1}(X)$ is defined using the recursive formula

$$(24) \quad u^0 := \begin{array}{c} \text{Diagram with two vertical lines.} \end{array} \quad u^p := \begin{array}{c} \text{Diagram with a box labeled } u^{p-1} \text{ connected to } p \text{ lines.} \end{array} + (-1)^p \begin{array}{c} \text{Diagram with a box labeled } u^{p-1} \text{ connected to } p \text{ lines, with a crossing on the right.} \end{array}.$$

Theorem 6.3. *The map $\mathcal{E}: \text{gr } C^D(M; X, \partial^*) \longrightarrow C^D(M; X, \partial^*)$ is a natural isomorphism of filtered complexes. In particular, there is a natural isomorphism*

$$(25) \quad H_n^D(M; X, \partial^*) \cong \bigoplus_{p+q=n} \widehat{H}_{q-2}(M; X, \partial^*) \otimes C_p^N(X)$$

for any spindle (X, \star) .

Proof. Naturality of \mathcal{E} is clear from the way it is constructed, and we shall show it is a chain map. First, observe that we can pull u^p through a line:

$$(26) \quad \begin{array}{c} \text{Diagram with a box labeled } u^p \text{ connected to } p \text{ lines, with a crossing on the left.} \end{array} = \begin{array}{c} \text{Diagram with a box labeled } u^p \text{ connected to } p \text{ lines, with a crossing on the right.} \end{array}.$$

This follows directly from (24) by induction on the number of strands. In particular, we can pull the right most line to the left either over or below u^{p-1} in the formula (24). Then a simple induction shows each component \mathcal{E}^p is a chain map:

$$\begin{aligned} \partial^* \mathcal{E}^p &= \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^{p-1} \text{ box.} \end{array} + (-1)^p \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^{p-1} \text{ box, with a crossing.} \end{array} \\ &= \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^{p-1} \text{ box.} \end{array} + (-1)^n \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^{p-1} \text{ box, with a crossing.} \end{array} + (-1)^p \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^{p-1} \text{ box, with a crossing.} \end{array} \\ &= \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^p \text{ box.} \end{array} + (-1)^n \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^{p-1} \text{ box, with a crossing.} \end{array} + (-1)^{n-1} \begin{array}{c} \text{Diagram with } \partial^* \text{ box and } u^{p-1} \text{ box, with a crossing.} \end{array} = \mathcal{E}^p \partial^*. \end{aligned}$$

Finally, the theorem follows from Corollary 6.2, because $\mathcal{E}^p(m \otimes \underline{x}) \in m \otimes \underline{x} + F^{p-1}$. \square

According to the theorem above every degenerate homology group splits into a direct sum of copies of whole homology groups in lower degrees. These in turn split canonically into normalized and degenerate parts and the latter can be recursively replaced with groups in lower degrees.

Corollary 6.4. *If the spindle (X, \star) is finite,*

$$(27) \quad H_n^D(X, \partial^\star) \cong \bigoplus_{p=1}^n \tilde{H}_{n-p}^N(X, \partial^\star)^{\oplus \frac{|X|}{1+|X|}(|X|^p - (-1)^p)}.$$

Proof. We proof the formula by induction on n . It is clear for $n < 2$, as both sides are trivial ($H_1^D(X, \partial^\star) = 0$, since $(x, x) = \partial^\star(x, x, x)$ for any $x \in X$). For bigger n notice first that $\text{rk } C_p^N(X) = |X|(|X| - 1)^{p-1}$. In particular, $\text{rk } C_{p+1}^N(X) = (|X| - 1) \text{rk } C_p^N(X)$ and Theorem 6.3 implies

$$(28) \quad \begin{aligned} H_{n+1}^D(X, \partial^\star) &\cong \tilde{H}_{n-1}^N(X, \partial^\star)^{\oplus |X|} \oplus \bigoplus_{p=2}^n \tilde{H}_{n-p}^N(X, \partial^\star)^{\oplus |X|(|X|-1)^{p-1}} \\ &\cong \tilde{H}_{n-1}^N(X, \partial^\star)^{\oplus |X|} \oplus H_{n-1}^D(X, \partial^\star)^{\oplus |X|} \oplus H_n^D(X, \partial^\star)^{\oplus (|X|-1)}. \end{aligned}$$

Using induction we show that $\tilde{H}_{n-1}^N(X, \partial^\star)$ does not appear in the second nor the third term, and $\tilde{H}_{n-2}^N(X, \partial^\star)$ comes only from the last summand in multiplicity $|X|(|X| - 1)$. For $p > 2$, the group $\tilde{H}_{n+1-p}^N(X, \partial^\star)$ appears with multiplicity

$$(29) \quad \frac{|X|}{1+|X|} \left(|X|(|X|^{p-2} - (-1)^p) + (|X| - 1)(|X|^{p-1} + (-1)^p) \right) = \frac{|X|}{1+|X|} (|X|^p - (-1)^p)$$

as desired. \square

The above corollary shows that the size of $H^D(X)$ is determined by $H^N(X)$ and the size of the spindle X . In fact, $H^D(X)$ determines X unless $\tilde{H}^N(X) = 0$.

Proposition 6.5. *Suppose $\tilde{H}_p^N(X, \partial^\star) \neq 0$ for some p . Then a spindle homomorphism $\varphi: X \rightarrow X'$ induces an isomorphism on degenerate homology if and only if it is an isomorphism of spindles.*

Proof. Choose the smallest p for which $\tilde{H}_p^N(X, \partial^\star) \neq 0$. Then

$$(30) \quad H_{p+1}^D(X, \partial^\star) \cong \tilde{H}_p^N(X, \partial^\star) \otimes C_0^N(X, \partial^\star)$$

and the map induced by φ has the form

$$(31) \quad \varphi_*: H_{p+1}^D(X, \partial^\star) \rightarrow H_{p+1}^D(X', \partial^\star), \quad (\dots) \otimes x \mapsto (\dots) \otimes \varphi(x).$$

Hence, φ must be bijective if φ_* is an isomorphism. \square

7. DEGENERATE GENERALIZED RACK HOMOLOGY

In the case of the rack differential $\partial^R := \partial^+ - \partial^*$ the degenerate complex is no longer isomorphic to $\text{gr } C^D(M; X)$. Instead we shall show it is isomorphic to the total complex of the bicomplex $B_{pq}(M; X) := \widehat{C}_{q-2}(M; X) \otimes C_p^N(X)$. We filter the bicomplex $B(M; X)$ by columns, i.e. $\mathcal{F}^p B := \widehat{C}(M; X) \otimes C_p^N(X)$.

Due to Corollary 5.2, the chain modules $B_{pq}(M; X)$ are isomorphic to quotients $\mathcal{F}_{p+q}^p / \mathcal{F}_{p+q}^{p-1}$, which can be seen as subgroups of $C_{p+q}^D(M; X)$. More precisely, there is a family of monomorphisms $i_{pq}: B_{pq}(M; X) \longrightarrow C_{p+q}^D(M; X)$ defined as $i_{pq}(m \otimes \underline{x} \otimes \underline{y}) = m \otimes \underline{x} \otimes s(\underline{y})$ and visualized by thickening the p -th strand:

$$(32) \quad i_{pq} := (-1)^p \begin{array}{c} \text{diagram with } p \text{ thickened strands and } 0 \text{ normal strands} \end{array}.$$

Recall that doubling the left most element is coherent with the rack differential, i.e. $\partial^R s(\underline{x}) = s(\partial^R \underline{x})$. We now define a family of maps $\mathcal{E}^p: \mathcal{F}^p B \longrightarrow C_{p+q}^D(M; X)$ by composing $i_{p\bullet}$ with the map $u^p: C_{p+1}(X) \longrightarrow C_{p+1}(X)$ defined in the previous section and the splitting homomorphism $\alpha: C^N(X) \longrightarrow C(X)$ acting on the normalized factor:

$$(33) \quad \mathcal{E}^p := (-1)^p \begin{array}{c} \text{diagram with } p \text{ thickened strands, } u^p \text{ box, and } \alpha_p \text{ box} \end{array}.$$

The homomorphism α_p translates the normalized rack differential into the unnormalized one, which makes graphical calculus easier—we do not have to care about repetitions when applying u^p .

Theorem 7.1. *The map $\mathcal{E}: \text{Tot}(B(M; X)) \longrightarrow C^D(M; X)$ is an isomorphism of filtered complexes. In particular, if R is a p.i.d. there is a short exact sequence*

$$(34) \quad 0 \longrightarrow \bigoplus_{p+q=n} \widehat{H}_{q-2}(M; X, \partial^R) \otimes H_p^N(X, \partial^R) \longrightarrow H_n^D(M; X, \partial^R) \\ \longrightarrow \bigoplus_{p+q=n-1} \text{Tor} \left(\widehat{H}_{q-2}(M; X, \partial^R), H_p^N(X, \partial^R) \right) \longrightarrow 0$$

which splits.

Proof. As before, $\mathcal{E}^p(\underline{x}) \in \underline{x} + F^{p-1}$, which in the view of Corollary 6.2 guarantees \mathcal{E} is an isomorphism if it is a chain map. This follows immediately from Theorem 6.3. Indeed,

$$\partial^+ u^p = \begin{array}{c} \text{diagram with } \partial^+ \text{ box and } u^{p-1} \text{ box} \end{array} + (-1)^p \begin{array}{c} \text{diagram with } \partial^+ \text{ box and } u^{p-1} \text{ box, and a crossing} \end{array}$$

$$\begin{aligned}
&= \begin{array}{c} \text{Diagram 1: } u^{p-2} \text{ box over } \partial^* - \partial^+ \text{ box, } p \text{ strands, } 0 \text{ strand} \\ - (-1)^p \begin{array}{c} \text{Diagram 2: } u^{p-1} \text{ box, } p \text{ strands, } 0 \text{ strand} \\ + (-1)^p \begin{array}{c} \text{Diagram 3: } u^{p-1} \text{ box, } p \text{ strands, } 0 \text{ strand} \\ - (-1)^p \begin{array}{c} \text{Diagram 4: } u^{p-2} \text{ box over } \partial^+ - \partial^* \text{ box, } p \text{ strands, } 0 \text{ strand} \end{array} \end{array} \end{array} \\
&= \begin{array}{c} \text{Diagram 5: } u^{p-2} \text{ box over } \partial^+ - \partial^* \text{ box, } p \text{ strands, } 0 \text{ strand} \\ + (-1)^{p-1} \begin{array}{c} \text{Diagram 6: } u^{p-2} \text{ box over } \partial^+ - \partial^* \text{ box, } p \text{ strands, } 0 \text{ strand} \\ + (-1)^{p+1} \begin{array}{c} \text{Diagram 7: } u^{p-1} \text{ box over } d_0^+ - d_0^* \text{ box, } p \text{ strands, } 0 \text{ strand} \end{array} \end{array} = u^{p-1}(\partial^+ - \partial^*),
\end{array}
\end{aligned}$$

and together with $\partial^R s \alpha = s \alpha \bar{\partial}^R$ it implies²

$$\partial^R \mathcal{E}^p(a \otimes b) = \mathcal{E}^p(\partial^R a \otimes b) + (-1)^q \mathcal{E}^p(a \otimes \bar{\partial}^R b)$$

for $a \in \widehat{C}_q(M; X)$ and $b \in C_p^N(X)$. The existence of a short exact sequence follows from the Künneth theorem. \square

Corollary 7.2. *Suppose a spindle homomorphism $\varphi: X \rightarrow X'$ induces an isomorphism on normalized rack homology $\varphi_*: H^N(X, \partial^R) \rightarrow H^N(X', \partial^R)$. If M is an X' -module such that the induced map $\varphi_*: H^N(M^\varphi; X, \partial^R) \rightarrow H^N(M; X', \partial^R)$ is also an isomorphism, so is $\varphi_*: H^D(M^\varphi; X, \partial^R) \rightarrow H^D(M; X', \partial^R)$.*

Proof. Use a similar induction argument to the one from the proof of Theorem 5.8 to show that $B(M^\varphi; X) \cong B(M; X')$. \square

Remark 7.3. The proof can be easily extended to the case $\partial^{a,b} = a\partial^* + b\partial^+$, when one replaces $B(M; X)$ with the bicomplex $\widehat{C}(M; X, \partial^{a,b})[2] \otimes C^N(X, b \cdot \partial^R)$. In particular, $H^D(X, \partial^{a,b})$ is fully determined by the rack-type homology $H^N(X, b \cdot \partial^R)$, which can be computed from $H^N(X, \partial^R)$; see Lemma 5.3 from [PP] for a more detailed statement.

APPENDIX A. SPECTRAL SEQUENCES

This section provides basic definitions and results on spectral sequences. Most theorems are left without proves, which can be found for example in [M]. The exception is Lemma A.8, which seems to be unknown. It is a key ingredient to the proof of Theorem 5.8.

Definition A.1. A (homological) spectral sequence is a sequence of bigraded R -modules $\{E^r\}_{r \in \mathbb{N}}$ together with differentials $d^r: E^r \rightarrow E^r$, each of degree $(-r, r-1)$, such that $H_*(E^r, d^r) \cong E^{r+1}$. The chain complex (E^r, d^r) is called the r -th page of the spectral sequence (E, d) .

² For clarity we used here $\bar{\partial}^R$ for the rack differential in the normalized complex, to distinguish it from the unnormalized one.

Definition A.2. A *morphism of spectral sequences* $f: E \rightarrow \bar{E}$ is a sequence of module homomorphisms $f^r: E^r \rightarrow \bar{E}^r$ such that f^{r+1} is equal to the induced homomorphism $f_*: H(E^r, d^r) \rightarrow H(\bar{E}^r, d^r)$.

We shall be interested only in the so called *first quadrant* spectral sequences, i.e. those with $E_{pq}^r = 0$ if $p < 0$ or $q < 0$. Then it must be $E_{pq}^r = E_{pq}^{r+1}$ for $r > p + q + 1$, which means the sequence *converges*; we shall write E_{pq}^∞ for the limit groups. A natural source of such spectral sequences is provided by filtered chain complexes.

Definition A.3. A *filtration* of a graded R -module M is a sequence of graded submodules $\mathcal{F}^0 M \subset \mathcal{F}^1 M \subset \mathcal{F}^2 M \subset \dots$ such that $\bigcup_{i \in \mathbb{N}} \mathcal{F}^i M = M$. We say the filtration is *bounded* if for every $n \in \mathbb{Z}$ there exists $i \in \mathbb{N}$ such that $M_n = \mathcal{F}^i M_n$.

Given a filtered graded R -module (M, \mathcal{F}) we define its *graded associated module* as

$$(35) \quad \text{gr } M := \bigoplus_{i \in \mathbb{N}} \mathcal{F}^{i+1} M / \mathcal{F}^i M.$$

Definition A.4. A spectral sequence $\{E^r, d^r\}_{r \in \mathbb{N}}$ *converges* to a filtered module M if it stabilizes and $E_{pq}^\infty \cong \text{gr}_p M_q$.

Given a filtered chain complex (C, \mathcal{F}) we filter its homology by images of homology of the subcomplexes: $\mathcal{F}^p H(C) := \text{im}(H(\mathcal{F}^p C) \rightarrow H(C))$. Clearly, $\mathcal{F}H$ is bounded if so is $\mathcal{F}C$.

Theorem A.5 (cf. [M]). *Let (C, \mathcal{F}) be a chain complex with a bounded filtration. Then there exists a spectral sequence (E^r, d^r) with $E_{pq}^1 = H_{p+q}(\mathcal{F}^p C / \mathcal{F}^{p-1} C)$ converging to $H(C)$.*

Given filtered chain complexes (C, \mathcal{F}) and (C', \mathcal{F}') we say a chain map $f: C \rightarrow C'$ is *filtered* if $f(\mathcal{F}^p C) \subset \mathcal{F}^p C'$ for any p . Such a map induces a morphism on the associated quotients $f_*: \mathcal{F}^p C / \mathcal{F}^{p-1} C \rightarrow \mathcal{F}^p C' / \mathcal{F}^{p-1} C'$ and a homomorphism of spectral sequences $f^r: E^r \rightarrow \bar{E}^r$ (the associated quotients form the 0-th page).

Theorem A.6 (cf. [M]). *Assume $f: (C, \mathcal{F}) \rightarrow (C', \mathcal{F}')$ is a filtered chain map and the filtrations on C and C' are bounded. Denote by E^r and \bar{E}^r the spectral sequences for (C, \mathcal{F}) and (C', \mathcal{F}') respectively. If $f_{pq}^\infty: E_{pq}^\infty \rightarrow \bar{E}_{pq}^\infty$ is an isomorphism for all $p + q = n$, then so is $f_*: H_n(C) \rightarrow H_n(C')$.*

In other words, if f induces an isomorphism on graded modules associated to homology, then the map of homology is also an isomorphism. In fact, it is enough to have an isomorphism at certain page to deduce that homology of C and C' are isomorphic.

Corollary A.7. *In the situation as above assume that $f^r: E^r \rightarrow \bar{E}^r$ is an isomorphism for some $r \in \mathbb{N}$. Then $f_*: H(C) \rightarrow H(C')$ is an isomorphism.*

However, we need a weaker result with f^r being an isomorphism only in certain degrees.

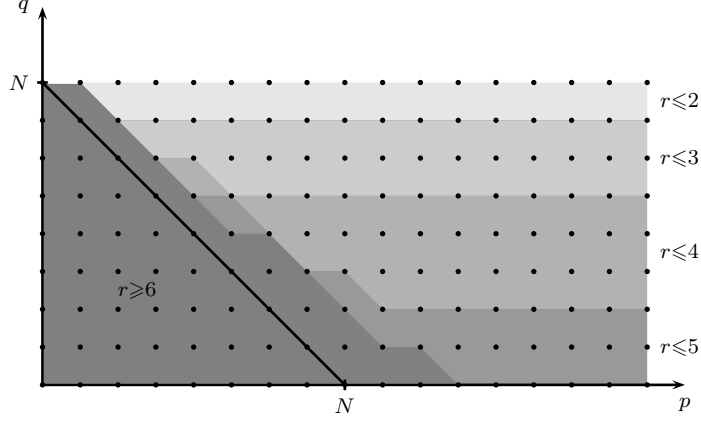


FIGURE 2. The sets \mathcal{I}^r of points (p, q) at which f_{pq}^r is an isomorphism, visualized for $N = 8$. A region labeled $r \leq i$ belongs to every \mathcal{I}^r with $r \leq i$. The skew line represents the diagonal $p + q = N$

Lemma A.8. *Assume there is a homomorphism of spectral sequences $f_{pq}^r: E_{pq}^r \rightarrow \bar{E}_{pq}^r$ such that f_{pq}^2 is an isomorphism for $q \leq N$. Then $f_{pq}^\infty: E_{pq}^\infty \rightarrow \bar{E}_{pq}^\infty$ is an isomorphism for $p + q = N$.*

The main idea is that if a chain map $g_i: C_i \rightarrow C'_i$ between two chain complexes is an isomorphism for $i = i_0, i_0 - 1$, and $i_0 + 1$, then $g_*: H_i(C) \rightarrow H_i(C')$ is an isomorphism. Indeed, let $t_i C$ be the subquotient of C with $(t_i C)_j = C_j$ whenever $j \in \{i - 1, i, i + 1\}$ and zero otherwise. Likewise we define $t_i C'$. Then $H_i(t_i C) \cong H_i(C)$ and similarly for C' , whereas $g: C \rightarrow C'$ descends to an isomorphism between $t_i C$ and $t_i C'$.

Therefore, as f_{pq}^2 is an isomorphism for $q \leq N$, and the differential d^2 increases q by 1 and decreases p by 2, f_{pq}^3 is an isomorphism whenever $q \leq N - 1$ or $q = N$ and $p < 2$. The next differential, d^3 , increases q by 2 and decreases p by 3. Hence, f_{pq}^4 is an isomorphism for $q \leq N - 3$ and a few more points: $p < 2$ and $q \leq N$, or $p = 2$ and $q \leq N - 1$, or $p = 3, 4$ and $q \leq N - 2$. By repeating this process we see, that the set of points (p, q) for which f_{pq}^r is an isomorphism form a staircase diagram, see Fig. 2. We want to show that the whole diagonal $p + q = N$ is below the stairs for any r .

Proof of Lemma A.8. Define functions $p^i(r) := \sum_{k=1}^i (r - k)$. In particular, we have

$$(36) \quad p^i(r + 1) = p^i(r) + i \quad p^{i+1}(r + 1) = p^i(r) + r.$$

Let $\mathcal{I}^r \subset \mathbb{N} \times \mathbb{N}$ consists of pairs (p, q) satisfying the following condition:

$$(37) \quad \begin{cases} q \leq N, & \text{if } p = 0, \\ q \leq N + i - p, & \text{if } p^{i-1}(r) < p \leq p^i(r) \text{ for some } i, \\ q \leq N + (r - 2) - p^{r-2}(r), & \text{if } p > p^{r-2}(r). \end{cases}$$

Notice that all \mathcal{I}^r contain the diagonal $p + q = N$, and f_{pq}^2 is an isomorphism for $(p, q) \in \mathcal{I}^2$. We shall show by induction that f_{pq}^{r+1} is also an isomorphism whenever $(p, q) \in \mathcal{I}^{r+1}$. For

that we have to check that f_{pq}^r , $f_{p+r, q-r+1}^r$, and $f_{p-r, q+r-1}^r$ are isomorphisms, as d^r lowers p by r and increases q by $r-1$.

- Since $p^i(r) < p^i(r+1)$, we have $\mathcal{I}^{r+1} \subset \mathcal{I}^r$ and $(p, q) \in \mathcal{I}^r$. Hence, f_{pq}^r is an isomorphism.
- Clearly $f_{p+r, q-r+1}^r$ is an isomorphism if $q < r-1$ or $p > p^{r-1}(r+1)$ (the last case in 37). Assume then that $p^i(r+1) \leq p < p^{i+1}(r+1)$ for some i . Then $p+r > p^{i+1}(r+1) = p^i(r) + r > p^{i+1}(r)$, and $(p+r) + (q-r+1) = p+q+1 \leq N+i+1$, so that $(p+r, q-r+1) \in \mathcal{I}^r$.
- Finally, $f_{p-r, q+r-1}^r$ is an isomorphism if $p < r$, so that we can assume $p \geq p^i(r+1)$ for some $i > 0$ (as $p^1(r+1) = r$). There are two subcases to check.
 - *Case 1:* $p > p^{r-1}(r+1)$. We have

$$q+r-1 \leq N+2(r-1)-p^{r-1}(r+1) = N+(r-2)-p^{r-2}(r),$$

which guarantees $(p-r, q+r-1) \in \mathcal{I}^r$.

- *Case 2:* $p^i(r+1) < p \leq p^{i+1}(r+1)$ for some $i = 0, \dots, r-1$ (case $i = 0$ happens if $p = r$). Then $p^{i-1}(r) < p-r \leq p^i(r)$ and since

$$(p-r) + (q+r-1) = p+q-1 \leq N+i,$$

we have $(p-r, q+r-1) \in \mathcal{I}^r$.

Hence, f_{pq}^{r+1} is an isomorphism if $(p, q) \in \mathcal{I}^r$, and so is f_{pq}^∞ for $p+q \leq N$. \square

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